1701. Separate the variables in the following differential equation, writing it in the form $f(y)\frac{dy}{dx} = g(x)$ for some functions f and g:

$$x^2\frac{dy}{dx} - 2\sin x\cos y = \sin x.$$

1702. A quadratic function g has $g(a) = g(b)$, for some $a \neq b$. Prove that $g'(a) + g'(b) = 0$.

1703. Find y in simplified terms of x, if

(a) $\ln y = 2\ln x - \ln\sqrt{x}$,

(b) $\log_2 y = 3\log_2 x + \log_4 x$.

1704. A student writes: "When you step on the pedals of a bicycle, a frictional force is generated, which acts forwards on the back wheel of the bicycle." Explain whether this is correct.

1705. Three sequences are given as follows:

$$\begin{aligned} a_n &= 1, 3, 5, 7, ... \\ b_n &= 2, 4, 6, 8, ... \\ c_n &= 2, 12, 30, 56, ... \end{aligned}$$

(a) Give ordinal formulae for a_n and b_n.

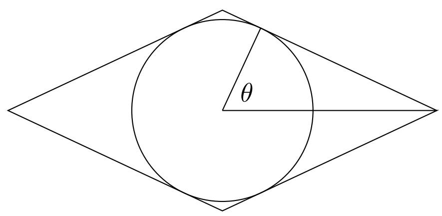
(b) Express c_n in terms of a_n and b_n.

(c) Hence, find the first term c_n to exceed 1000.

1706. Let $a, b, c, d > 0$ be distinct constants. Solve the following equation:

$$\frac{(x^4 - a^4)(x^4 + b^4)}{(x^4 - c^4)(x^4 + d^4)} = 0.$$

1707. A rhombus, with perimeter 16, is tangent to a unit circle at four points. The radius shown below is to one of those points.



(a) Show that the side length of the rhombus may be expressed as $\tan\theta + \cot\theta$.

(b) Hence, show that $\tan^2\theta - 4\tan\theta + 1 = 0$.

(c) Hence, find the interior angles of the rhombus.

1708. A student makes the following claim:

"If function f has the property that, for all $x \in \mathbb{R}$, $f(x + 1) > f(x)$, then f is increasing for all $x \in \mathbb{R}$."

Explain why f, defined by $f(x) = \sin(2\pi x) + x$, is a counterexample to this claim.

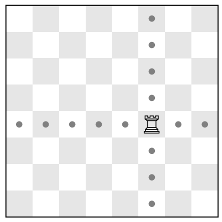
1709. It is given that $\int_0^1 f(x)\,dx = 2$. Evaluate

(a) $\int_0^1 1 + f(x)\,dx$,

(b) $\int_0^1 3\big(x^2 + f(x)\big)\,dx$.

1710. In the equation $16^x - 8^x + 4^x - 2^x = 0$, factorise the left-hand side fully, and hence show that the solution is $x = 0$.

1711. In chess, a rook threatens squares as shown. Find the probability that, if two rooks are placed on a chessboard at random, they threaten each other.



1712. Show that $y = 2^x$ may be transformed to $y = -\frac{1}{2^x}$ by a rotation.

1713. Let x, y, z be consecutive natural numbers. Prove that $x^3 + y^3 + z^3$ is divisible by three and by at least one of x, y, z.

1714. "The line $y = x$ is normal to the cubic $y = x^3 - x$." True or false?

1715. A rectangle is being transformed by stretches in both of its dimensions. At a particular time, these dimensions are 4×6, in cm, and they are changing at rates of -2 and 3 cm per second respectively. Find the instantaneous rate of change of the area.

1716. Points $(1, -3)$, $(-3, 1)$, $(2, 6)$ lie on a parabola. Find its equation, in the form $y = ax^2 + bx + c$.

1717. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

① $\sqrt[3]{xy} = z$,

② $xy = z^3$.

1718. A function f is defined by

$$f(x) = 8e^x(1 - 2e^x).$$

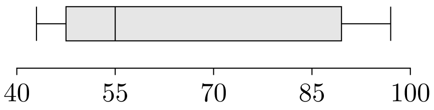
Show that the stationary value is $f(x) = 1$.

1719. A projectile is launched at 10 ms^{-1}, at 30° above the horizontal. Show that raising the launch point to 1 metre above ground level will add around 1.5 metres to the range.

1720. Three dice are rolled. State, with a reason, which, if either, of the following events has the greater probability:

① two fives and a six, in any order,
② a four, a five and a six, in any order.

1721. The marks in a test, for a year group of 200 pupils, are summarised in a box-and-whisker diagram:



State, with a reason, if the following are definitely true of this set of data:

(a) Exactly half the scores are greater than or equal to the median.
(b) The mode is lower than the median.
(c) The mean is higher than the median.

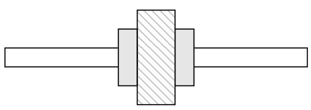
1722. A positive, monic, cubic function g has $x = 2$ as a fixed point, a stationary point, and a point of inflection.

(a) Explain which of the above facts are needed to show that $\text{g}(x) = (x-2)^3 + c$.
(b) Determine the value of c.

1723. Solve $(\cos x - 2)(2\cos 2x - 1) = 0$, for $0 \leq x \leq 2\pi$.

1724. You are given that, for variables p and q, $q = p^2 + 1$. Using a graphical method, or otherwise, show that $p + 2q > 1$.

1725. A carpenter's vice, attached to a workbench, is made of two adjustable jaws, shaded solid in the diagram below, which are clamped around a block of wood.



The block of wood has mass 5 kg, and each jaw of the vice can exert a maximum horizontal force of 500 N. Determine the range of possible values of μ, the coefficient of friction between the jaws and the wood.

1726. Prove that every hexagon must have at least one exterior angle β satisfying $\beta \leq \frac{\pi}{3}$ radians.

1727. A sequence is given, for constants $a, b > 0$, by $u_{n+1} = au_n + b, \quad u_1 = 1$. Prove that this sequence is neither arithmetic nor geometric.

1728. Solve $\dfrac{5\sqrt{x}}{2\sqrt{x}+1} - \dfrac{2\sqrt{x}-1}{\sqrt{x}} = \dfrac{1}{2}$.

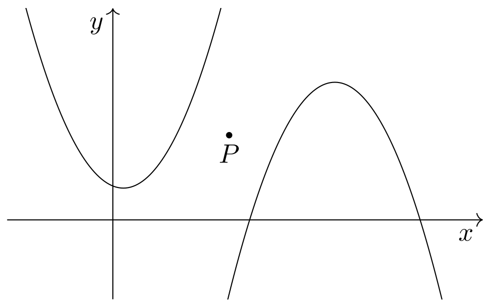
1729. A student is performing a binomial hypothesis test concerning a potentially biased selection process. The student writes "$H_0 : x = 10$, where x is the number of jobs out of the twenty in the sample which went to women." Explain why this cannot be a correct formulation of a statistical hypothesis, and give a corrected version.

1730. Give the range of each of the following functions, over the domain $[-2, 2]$.

(a) $x \mapsto x^2 + 1$,
(b) $x \mapsto x^3 + 1$,
(c) $x \mapsto x^4 + 1$.

1731. A set of four lines forms a square. The first three are $y = \sqrt{3}x$, $\sqrt{3}y = -x$ and $\sqrt{3}x + 3y = 5$. Find the two possible equations of the last line.

1732. The graphs $y = x^2 - ax + b$ and $y = -x^2 + cx + d$ may be transformed onto each other by a rotation 180° around point P.



Show that the coordinates of point P are

$$\left(\frac{a+c}{4}, \frac{4(b+d) - a^2 - c^2}{8}\right).$$

1733. Prove the following implication, in which f and g are polynomial functions:

$$\int_0^x \text{f}(t)\,dt \equiv \int_0^x \text{g}(t)\,dt \implies \text{f}(x) \equiv \text{g}(x).$$

1734. Let $\text{f}(x) = 8x^3 + 14x^2 - 553x - 1224$.

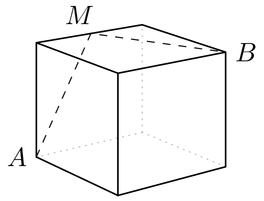
(a) Explain why any fixed point of the function
$$\text{g}(x) = \frac{8x^3 + 14x^2 - 1224}{553}$$
must be a root of $\text{f}(x) = 0$.
(b) Run the iteration $x_{n+1} = \text{g}(x_n)$, with $x_0 = 0$, to find a root of $\text{f}(x) = 0$.
(c) Using this root, factorise $\text{f}(x)$ fully.

1735. Express the following set as a list of elements, in the form $\{a_1, a_2, ..., a_n\}$.

$$\{x \in \mathbb{N} : \log_2 x < 5\} \cap \{x \in \mathbb{N} : \log_4 x > \tfrac{12}{5}\}.$$

1736. A jar contains m red and n yellow counters. From the jar, two counters are taken out at random. Find, in terms of m and n, the probability that these are one of each colour.

1737. The diagram shows a solid cube, with the shortest path between two vertices marked.



Determine angle AMB.

1738. Consider $(y - |x| + 1)(y + |x| - 1) \leq 0$.

(a) Explain, referring to the graphs $y - |x| + 1 = 0$ and $y + |x| - 1 = 0$, why the boundary of the (x, y) region defined by this inequality consists of four line segments.

(b) On a sketch, shade the (x, y) region which is satisfied by the inequality.

1739. The interior angles of an irregular pentagon form an AP. Give, in radians, the upper bound on the value of

(a) the largest angle,

(b) the common difference.

1740. Projectiles are launched on Earth and on the Moon, from ground level, with the same initial speed u and angle of inclination θ. On the Moon, the acceleration due to gravity is around $\frac{1}{6}g$ ms^{-2}. Prove that, if the range of the projectile on Earth is d, then the range on the Moon is around $6d$.

1741. Express $12x^2 - 2x - 13$ in terms of $X = 2x - 1$, giving your answer in the form $aX^2 + bX + c$.

1742. A variable X has distribution $B(10, 0.5)$.

(a) Explain, without doing any calculations, why $\mathbb{P}(X = a) = \mathbb{P}(X = 10 - a)$ for any a.

(b) Hence, write down the value of

$$\mathbb{P}\left(X = 2 \mid X \in \{2, 8\}\right).$$

1743. Find the linear function g for which

$$\int_0^1 \mathrm{g}(y)\,dy = 1, \quad \mathrm{g}(0) = 0.$$

1744. Factorise $a^3 - b^3$, using the fact that setting $a = b$ makes the expression zero.

1745. An isosceles triangle has perimeter 18 and area 12.

(a) By defining suitable variables on a diagram, show that this information may be expressed in the simultaneous equations

$$2a + b = 18$$
$$b\sqrt{4a^2 - b^2} = 48.$$

(b) Show that $b^3 - 9b^2 + 64 = 0$.

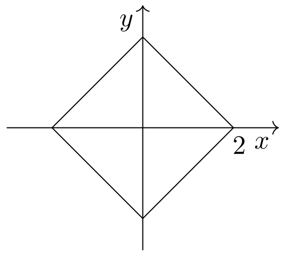
(c) Hence, find the side lengths of the triangle, given that they are integers.

1746. You are given that

$$y = (1 + \cos x)(2 - \cos x).$$

Find a simplified expression for $\dfrac{dy}{dx}$.

1747. The locus of the equation $|x| + |y| = 2$ is a square:



Determine which of the points $(3, 2)$ and $(4, 0)$ is closer to the square.

1748. Write the following in terms of 3^x:

(a) 3^{2x},

(b) 3^{2x-1},

(c) 3^{1-2x}.

1749. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$xy \geq 1, \qquad -1 < x - y < 1.$$

1750. A parabola is given as $y = x^2 + x$. Tangents are drawn to the parabola at $x = \pm k$, for some $k > 0$.

(a) Show that these tangents have equations

$$y - (k^2 + k) = (2k + 1)(x - k),$$
$$y - (k^2 - k) = (-2k + 1)(x + k).$$

(b) Show that the tangents meet on the y axis.

1751. Prove that the diagonals of a rhombus have lengths d_1, d_2 given by $d_1, d_2 = a\sqrt{2 \pm 2\cos\beta}$, where a is the side length and β is any interior angle.

1752. You are given that the variables x, y, z satisfy

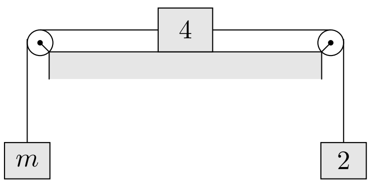
$$z = ax + by + c,$$

for constants a, b. Determine whether each of the following statements is true:

(a) z and x are linearly related,

(b) if x and y are linearly related, then z and x are linearly related.

1753. A pulley system of three blocks is set up on a table as shown. Masses are given in kg. The coefficient of friction between the 4 kg block and the table is $\frac{1}{2}$. The system is in equilibrium.



(a) State two assumptions which are necessary to find the range of possible values of m.

(b) Explain we don't need to assume that the strings are inextensible.

(c) Making the necessary assumptions, find, in set notation, all possible values of m.

1754. In this question, do not use a calculator.

Evaluate exactly $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 6x - 6\sin x \, dx$.

1755. At meetings of a friendly society, everyone shakes hands with everyone.

(a) Find the total number of handshakes when 10 people are present.

(b) Find the total number of people present if there are 1485 handshakes.

1756. The line $2x + 3y + 5 = 0$ is closest to $(3/2, 6)$ at point Q. Find the coordinates of point Q.

1757. Show that, if the relationship between x and y is polynomial, of the form $y = bx^n$, then $\ln y$ and $\ln x$ are related linearly.

1758. The numbers $1, 2, 3, 4, 5, 6$ are randomly assigned, each to a different vertex of a regular hexagon. Find the probability that the numbers appear in either ascending or descending order around the perimeter.

1759. A convex quadrilateral has vertices at $(0, 0)$, $(1, 3)$, $(9, 8)$, and $(6, 1)$. Find its area.

1760. Take $g = 10$ in this question.

A piloted glider of mass m is being accelerated by a cable attached to a winch. The tension in the cable is modelled as a constant $\frac{1}{4}mg$. The glider starts from rest, 100 metres from the winch, and experiences lift proportional to its horizontal speed v_x, given by $L = 0.5mv_x$.

(a) Making any necessary assumptions, show that the glider takes off at 20 ms^{-1}.

(b) Determine the distance from the winch at which this happens.

1761. Solve for x in $\displaystyle\sum_{r=1}^{\infty} x^r = \frac{1}{2}$.

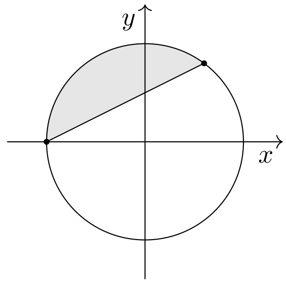
1762. State, with a reason, whether the following holds: "In any two-tail test for the mean of a normal distribution, the acceptance region is symmetrical about the hypothesised population mean μ."

1763. You are given that $\mathrm{f}'(x) = \mathrm{g}(x)$ for all x, and that the equation $\mathrm{f}(x) = 0$ has roots $x = \alpha$ and $x = \beta$. Evaluate the following:

(a) $\displaystyle\int_{\alpha}^{\beta} \mathrm{g}(x) \, dx$.

(b) $\displaystyle\int_{\alpha}^{\beta} \frac{2x + \mathrm{g}(x)}{3} \, dx$.

1764. On the circle $x^2 + y^2 = 1$, a chord is drawn from $(3/5, 4/5)$ to $(-1, 0)$, producing a minor segment, shaded below, of area A.



Show that $A \approx 0.71$.

1765. The positions of two particles, for $t \geq 0$, are given by $x_1 = t^2 - 3t$ and $x_2 = 4 - t^2$.

(a) Explain why $|2t - 3|$ gives the speed of the first particle, and find a similar expression for the speed of the second.

(b) Determine the set of t values for which the first particle is moving faster than the second.

1766. A quartic is defined by $y - k = kx^4 - kx^2$, for some constant $k \neq 0$. Show that this curve has stationary points, two of which have the same y coordinate.

1767. On any given day in winter, the probability that I wear a jacket is 83%, and the probability that it rains at some stage is 40%. On days when it rains at some stage, the probability that I wear a jacket is 95%.

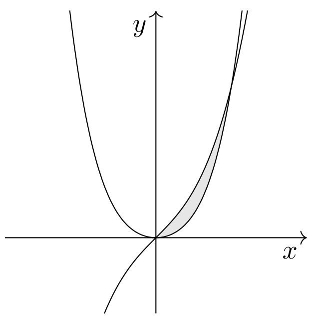
(a) Draw a tree diagram, conditioned on rain/no rain, to represent this information.

(b) Find the probability that I wear a jacket and it doesn't rain.

(c) Determine the probability that it rains at some stage, given that it is a day on which I am wearing a jacket.

1768. A right-angled triangle has side lengths (a, b, c), where $a, b, c \in \mathbb{N}$. Prove by contradiction that this triangle is not isosceles.

1769. The diagram shows a region enclosed by the curves $y = x^4 + x^2$ and $y = x^3 + x$.



Show that the area of the shaded region is $\frac{13}{60}$.

1770. A slate of mass m kg is attached to a roof inclined at $\theta°$ to the horizontal. Give the magnitudes of the following forces on the slate, in terms of m, g, θ:

(a) the contact force parallel to the roof,

(b) the contact force perpendicular to the roof,

(c) the total contact force.

1771. The pair of simultaneous equations $2x + ay = 3$ and $4x = 2y + b$ has infinitely many points (x, y) which satisfy it. Find a and b.

1772. The ellipse $3x^2 + y^2 = 1$ is transformed to a second ellipse $(x + 2)^2 + 3(y - 2)^2 = 1$ by reflection in a straight line. Find the equation of the line.

1773. Functions f and g are given, for constants a, b, by

$$\begin{aligned} \mathrm{f}(x) &= 4x^2 - 10x + a, \\ \mathrm{g}(x) &= x^2 + 6x + b. \end{aligned}$$

The solution set of the inequality $\mathrm{f}(x) \geq \mathrm{g}(x)$ is $(-\infty, -2/3] \cup [6, \infty)$. Determine the value of $b - a$.

1774. A researcher is analysing historical data, looking at correlation between extreme weather events and incidences of civil unrest. He logs the numbers w_i and c_i of each in five randomly selected years.

w_i	73	30	19	84	66
c_i	6	4	4	12	11

He then sets up a hypothesis test, using critical values from the following entry in the usual table:

1-tail	5%	2.5%	1%	0.5%
2-tail	10%	5%	2%	1%
$n = 5$	0.8054	0.8783	0.9343	0.9587.

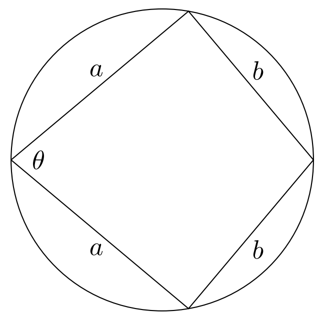
(a) Using the statistical facility on your calculator, verify that the sample correlation coefficient is approximately 0.82.

(b) Write down suitable hypotheses for the test, defining any variable you use.

(c) Carry out the test, at the 5% level, to ascertain whether or not there is evidence for correlation between the two variables.

(d) Such a test requires the assumption that the underlying population has a bivariate normal distribution. Comment on whether this is likely to be appropriate or not.

1775. Quadrilateral K is a kite, and it is cyclic. It has side lengths a and b, and the angle between the two sides of length a is θ.



Prove that the area of K is $A = \frac{1}{2}(a^2 + b^2)\sin\theta$.

1776. The points on a straight line are described, in terms of a parameter $t \in \mathbb{R}$, by a position vector

$$\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

Find the Cartesian equation of the line.

1777. Prove that, if two tangent lines are drawn to the parabola $y = 3x^2 - 5x + 2$ at $x = \pm a$, then they meet on the y axis.

1778. Solve the equation $\dfrac{1 + \frac{1}{x}}{1 - \frac{1}{x^2}} = x$.

1779. A random sample of ten data is taken from a large population, which is normally distributed. Using an appropriate definition for outliers, show that, in such a sample, the probability of there being at least one outlier is around 37%.

1780. A curve is given, for constants $a, b, c \neq 0$, as

$$y = \frac{ax + b}{cx}.$$

(a) Show that the curve may be written $y = p + \frac{q}{x}$, where $p, q \in \mathbb{Q}$ are constants to be determined.

(b) Hence, write down the two transformations that map the curve $y = \frac{1}{x}$ to this curve.

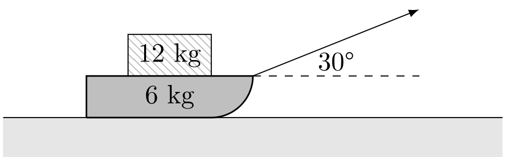
(c) Write down the equations of the horizontal and vertical asymptotes of the curve.

1781. Solve $3^{-2x} + 3^{-x} = 30$, giving any roots in the form $x = \log_a b$, for $a, b \in \mathbb{N}$.

1782. Show that the following is an identity, identifying all values of x for which the identity is undefined:

$$\frac{x^2 + 3x + 2}{3x^3 + 4x^2 + x} \equiv \frac{2}{x} - \frac{5}{3x + 1}.$$

1783. A sledge, carrying a 12 kg load of equipment, is being pulled along at constant velocity by means of a rope angled at 30° above the horizontal. The ground may be modelled as smooth; air resistance of magnitude 25 N is acting on the load.



(a) Draw force diagrams for the load and for the combined sledge and load.

(b) Find the tension in the rope.

(c) Find the least possible value of the coefficient of friction between the load and the sledge.

1784. Find a rational number in the interval $(\pi, \frac{101}{100}\pi)$.

1785. In each of the following, a list of quantities is given, which refers to a geometric series. State, with a reason, whether knowing the quantities in the list would allow you to calculate the sum.

(a) First term; last term; number of terms.

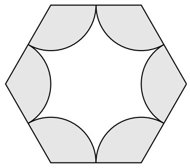
(b) Number of terms; mean.

(c) First term; last term; mean.

1786. Simplify $\frac{d}{dt}(\sqrt{x} + \sqrt[3]{y})$.

1787. Three lines have equations $y = 2x$, $y = 1$ and $y = 2$. Find the possible equations of a fourth line, if it is to form a rhombus with the first three.

1788. The diagram below shows a regular hexagon of side length 2 cm. All points which lie within 1 cm of a vertex have been shaded.



Find the proportion of the area of the hexagon which has been shaded.

1789. Given the distribution $X \sim N(0, 4)$, find

(a) $\mathbb{P}\big(X \in (1, 2)\big)$,

(b) $\mathbb{P}\big(X \in \{1, 2\}\big)$,

(c) $\mathbb{P}\big(X \in [1, 2]\big)$.

1790. A triangle has perimeter 9 and two angles $\arccos \frac{7}{8}$ and $\arccos \frac{11}{16}$. Solve to find the side lengths.

1791. Two unit vectors $\mathbf{a}$ and $\mathbf{b}$ can be written in terms of the standard perpendicular unit vectors as

$$\mathbf{a} = p\mathbf{i} + q\mathbf{j},$$
$$\mathbf{b} = q\mathbf{i} - p\mathbf{j}.$$

(a) Write down the value of $p^2 + q^2$.

(b) Express $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{a}, \mathbf{b}$ and p, q.

(c) Show that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

(d) Sketch $\mathbf{a}$ and $\mathbf{b}$, for $p = \frac{1}{2}$ and positive q.

1792. Evaluate $\displaystyle\lim_{x \to \frac{1}{3}} \frac{6x^2 + 13x - 5}{12x^2 - 7x + 1}$.

1793. From a large population, a sample of five is taken. Find the probability that all five lie between the quartiles of the population.

1794. Prove that, if a, b, c, d are in non-constant AP, then

$$\frac{d^2 - a^2}{c^2 - b^2} = 3.$$

1795. State whether each of the following implications hold, for $a, b \in \mathbb{R}$. If not, give a counterexample.

(a) $a > b \implies a^2 > b^2$,

(b) $a > b \implies -1 + 2a > -1 + 2b$,

(c) $a > b \implies {}^1\!/_a > {}^1\!/_b$.

1796. The graph $y = x^2 - 6x + k$, where k is a constant, crosses the x axis twice. The distance between these x intercepts is 2. Find k.

1797. Give a counterexample to the following claim: "A pentagon cannot have two pairs of parallel sides."

1798. True or false?

(a) $\displaystyle\sum_{i=a}^{c} u_i = \sum_{i=a}^{b} u_i + \sum_{i=b}^{c} u_i$,

(b) $\displaystyle\int_a^c y\,dx = \int_a^b y\,dx + \int_b^c y\,dx$.

1799. A six-sided die and an n-sided die, where $n > 6$, are rolled together. Determine, as a function of n, the probability that the score on the six-sided die is larger.

1800. Find $6 + 8 + 10 + ... + 2p$, in factorised terms of p, where p is a natural number greater than 2.

——— End of 18th Hundred ———